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FORCED OSCILLATIONS OF THREE-LAYERED ROD WITH A LIQUID INTERLAYER

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The problem of damping of three-layered rod oscillations by a in compressible Newtonian fluid which forms an interlayer between two ideally rectangular rods is considered.

Introduction. One means of vibrational protection of constructions involves the use of absorbing layers in the form of coatings or interlayers. The methods of calculation used [1-3] are based on the hypothesis of plane transverse cross-sections, which reduces the problem to a differential equation for the transverse displacement of a neutral plane of the rod. The employment of this hypothesis is possible in the case when the elasticity moduli of the layers are not strongly different and the bending of each layer is accompanied by pure shearing strains or tensile-compressive strains. Otherwise, one should allow for transverse displacements in a layer of a soft material. This necessity arises, for example, when studying active vibrational protection with an electrorheological damping layer [4, 5], where the formulas of [3] for a rigid damping layer are used to analyze the experiments.

To study the yield effect we consider oscillations of a pair of elastic rods with a Newtonian liquid interlayer.

Identical rods with length l are rectangular in cross-section with width b and height $2h$ ($h << b << l$). The rods are fastened at one end and between them there is a gap of width $2H$ filled by a viscous liquid. The other end of the obtained "sandwich" is free. We study the effect of the liquid in the gap on small forced oscillations of the sandwich which arise due to the effect of a periodic transverse force applied to the free end.

1. Liquid Dynamics. As is known [6], with bending of a rod, on different sides of the "neutral" surface, pure shearing strains originate on the convex side and compressive strains originate on the concave side. With bending of the sandwich, the surfaces bounding the liquid layer undergo deformations which are equal in value and opposite in sign. We select a system of coordinates with the origin on the "neutral" surface of the upper rod at the point of its attachment, the Z axis is directed along the undeformed rod, and the X axis along the smaller side of the cross section (upward). The length element dZ of the bases of both rods after deformation becomes equal to [6]

$$dZ' = dZ\,(1 \pm h/R)\,,$$

where the sign "+" refers to the lower and the sign "$-$" to the upper bases of the rods; R is the curvature radius of the surface $X = X(Z)$,

$$R \approx (d^2X/dZ^2)^{-1}\,.$$

The distance from the point of attachment to the point Z on the rod base after deformation becomes equal to

$$Z' = \int_0^Z \left(1 \pm h\frac{d^2X}{dZ^2}\right) dZ = Z \pm h\left(\frac{dX}{dZ} - \frac{dX}{dZ}(0)\right).$$

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In the considered case of an attached rod, dX/dZ at zero is equal to zero and

$$Z'(Z) = Z \pm h \frac{dX}{dZ}. \tag{1}$$

Thus, during bending the surfaces of the rods bounding the liquid layer move with local velocity

$$V(Z, t) = \pm h \frac{\partial}{\partial t} \left(\frac{dX}{dZ} \right). \tag{2}$$

To describe the flow of the liquid relative to the deformed rods we consider a liquid layer bounded by plane surfaces moving with a velocity which is variable along their length and which is determined by relations (2). When the characteristic time of bending oscillations of the sandwich, which in order of magnitude is equal to the reciprocal frequency of eigenoscillations of the rods ω_*^{-1}, is large compared to the characteristic time of the propagation of elastic disturbances in the liquid layer $\tau = l/c$ (c is the velocity of sound in liquid), i.e., $\omega_* \tau \ll 1$, the compressibility of the liquid may be ignored. We also neglect the inertial force in the liquid compared to the viscosity force, which is possible when $\rho_f V_0 l/\eta \ll 1$, where $V_0 = \max |V|$.

Introducing the system of coordinates x, y, z with the origin on the plane of sandwich attachment (in the center of the liquid layer), directing the X axis along the layer and the Z axis across (upward), we write the equations of plane liquid flow in the form

$$\frac{\partial P}{\partial x} = \eta \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial z^2} \right) = 0, \quad \frac{\partial P}{\partial z} = \eta \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial z^2} \right) = 0, \tag{3}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0. \tag{4}$$

The boundary conditions in the problem considered are the following. On the left motionless boundary of the layer the velocity of the liquid vanishes

$$v_x(x, 0, t) = v_z(x, 0, t) = 0. \tag{5}$$

On the free right boundary of the layer, the strains in the liquid are compensated by atmospheric pressure P_0

$$-Pn_i + \sigma_{ik} n_k = -P_0 n_i, \tag{6}$$

This means constancy of the normal and disappearance of the tangential stresses

$$-P + \sigma_{ij} n_i n_j = -P_0, \quad \sigma_{ij} n_i \tau_j = 0. \tag{7}$$

The tensor of viscous stresses σ is determined by the relation

$$\sigma_{ij} = \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right). \tag{8}$$

On the upper and lower surfaces adhesion conditions are satisfied which, according to (2), have the form (the coordinates z and Z of the two introduced systems coincide):

$$\begin{gathered} v_z(H, z, t) = V_0, \quad v_z(-H, z, t) = -V_0, \quad V_0 = h \frac{\partial}{\partial t} \left(\frac{\partial X}{\partial Z} \right), \\ v_x(H, z, t) = v_x(-H, z, t) = 0. \end{gathered} \tag{9}$$

By virtue of the problem symmetry

$$v_z(-x, z, t) = -v_z(x, z, t), \quad v_z(0, z, t) = 0.$$

Incompressible fluid flow in the direction x, perpendicular to the layer, arises due to the decrease and increase by equal values in the volumes of the layer portions on opposite sides of the "neutral" plane $x = 0$, i.e., it has the character of "overflowing." It is natural to assume that the velocity of overflowing possesses symmetry

$$v_x(-x, z, t) = v_x(x, z, t).$$

With allowance for the latter relations we find the velocity of the fluid in the form

$$v_z = \sum_{k=1}^{\infty} a_k(z, t)\left(\frac{x}{H}\right)^{2k-1}, \quad v_x = b_0(z, t) + \sum_{k=1}^{\infty} b_k(z, t)\left(\frac{x}{H}\right)^{2k}.$$

Satisfying the continuity equation (4) we find that the coefficients in this expansions are related by the expression

$$b_k = -\frac{H}{2k} a_k', \quad k = 1, 2, \ldots \tag{10}$$

(here and below the prime denotes the derivative with respect to z).

Differentiating the first and second equations in (3) with respect to z and x, respectively, eliminating pressure, and equating the coefficients with equal exponents x, we obtain with allowance for (10) for the coefficients of the equation

$$-H^3 b_0''' + 2H^2 a_1'' + 6a_2 = 0,$$

$$a_{k+2} = -\frac{1}{(2k+1)(2k+2)(2k+3)}\left(\frac{H^4}{2k} a_k^{\mathrm{IV}} + 2H^2(2k+1)\, a_{k+1}''\right). \tag{11}$$

If we omit small-scale flows, including the effect of the ends, from consideration and assume that the characteristic scale of the change in a_k is equal to the rod length l, then, discarding terms of the order of $(H/l)^2$ and higher, we take the subsequent coefficients of the expansion to be equal to zero. It follows from the condition $v_z(H) = V_0$ that $a_2 = V_0 - a_1$, and from the condition $v_x(H) = 0$ with allowance for (10) we have $b_0 = H(a_1' + V_0')/4$.

As a result Eq. (11) takes the form

$$a_1 = V_0 + \frac{1}{3} H^2 a_1'' - \frac{H^4}{24} a_1^{\mathrm{IV}} - \frac{H^4}{4} V_0^{\mathrm{IV}}.$$

Hence it follows that with the adopted accuracy $a_1 = V_0$, $a_2 = 0$, and the velocity of fluid flow is

$$v_z = V_0(z, t)\frac{x}{H}, \quad v_x = \frac{1}{2} V_0'(z, t)\, H\left[1 - \left(\frac{x}{H}\right)^2\right]. \tag{12}$$

The pressure distribution in the liquid layer is obtained by integrating the first equation in (3) with respect to x with allowance for (12):

$$P = P_0 - \eta\frac{x}{H} V_0' = P_0 - \eta h\frac{x}{H}\frac{\partial X''}{\partial t}. \tag{13}$$

2. Sandwich Oscillations. We proceed from the equations of dynamics of a rod slightly bent under the effect of an external force [6]

$$\rho_r S_r \frac{\partial^2 X}{\partial t^2} + EIX^{IV} = q(z, t) . \tag{14}$$

Here $I = hb^3/6$ is the inertial moment of the cross-section area of the rod relative to the axis y; $q = dF_x/dl$ is the normal component of the external force F per length unit of the rod. The effect of the liquid interlayer is caused by the inertia of its cross-section and also by the normal component of viscous forces on the surface of the rods. For the lower base of the upper plate it is determined by the relation

$$f_\eta/b = P(H) - P_0 - 2\eta \left.\frac{\partial v_x}{dx}\right|_{x=H} = \eta h V_0' = \eta h \frac{\partial X''}{\partial t} .$$

Thus, the equation of sandwich oscillations can be written in the form

$$(\rho_r S_r + \rho_f S_f/2) \frac{\partial^2 X}{\partial t^2} + EIX^{IV} - b\eta h \frac{\partial X''}{\partial t} = f(z, t) , \tag{15}$$

where f is the excitation force. We consider forced oscillations under the effect of a concentrated periodic force $F_0 \cos(\omega t)$ applied to the free end of the rod. We write the equation for these oscillations in dimensionless form introducing the scale of distance, the rod length l, and the time scale:

$$T = l^2 (\rho_r S_r/IE)^{1/2} .$$

Since only viscous force acts along the rod length, this equation has the form (the previous notation is preserved for dimensionless variables)

$$\delta^4 \frac{\partial^2 X}{\partial t^2} + X^{IV} - \nu \frac{\partial X''}{\partial t} = 0 , \tag{16}$$

where

$$\delta^4 = 1 + \frac{\rho_f H}{2\rho_r h} ; \quad \nu = \frac{\eta}{2} \left(\frac{S_r}{\rho_r IE} \right)^{1/2} .$$

The boundary conditions at the attached end ($z = 0$) are $X = 0$, $X' = 0$ and on the free end ($z = 1$) $X'' = 0$ and $X''' = f \cos(\omega t)$, where $f = F_0 l^3/EI$. The first condition on the free end expresses, as is known, the absence of force moments applied to it, and the second, the equality of the force of inner stresses to the force applied to the end.

We consider the steady-state oscillations of the sandwich. Seeking the solution in the form

$$X = \exp(i\omega t)\, \xi(z) ,$$

we obtain the equation for ξ

$$\delta^4 \omega^2 \xi + i\omega\nu\xi'' - \xi^{IV} = 0 . \tag{17}$$

The corresponding characteristic equation

$$\lambda^4 - i\omega\nu\lambda^2 - \omega^2\delta^4 = 0 \tag{18}$$

has the solution

$$\lambda^2 = \pm\, \omega\delta^2 \left(1 - \frac{\nu^2}{4\delta^4}\right)^{1/2} + i\omega\nu/2\,.$$

For real and imaginary components λ ($\lambda = \lambda' + i\lambda''$), hence we have

$$\lambda'^2 - \lambda''^2 = \pm\, \omega\delta^2 \left(1 - \frac{\nu^2}{4\delta^4}\right)^{1/2}, \quad \lambda'\lambda'' = \frac{\omega\nu}{4}\,. \tag{19}$$

The system (19) possesses the following properties. If the pair of numbers $\lambda' = r_1$, $\lambda'' = r_2$ are roots, then the pairs of numbers $\lambda' = -r_1$, $\lambda'' = -r_2$; $\lambda' = r_2$, $\lambda'' = r_1$, and $\lambda' = -r_2$, $\lambda'' = -r_1$ are also roots of Eqs. (19). Consequently, the solution of the characteristic equation can be written in the form

$$\lambda_1 = \alpha + i\beta\,, \;\; \lambda_2 = -\alpha - i\beta\,, \;\; \lambda_3 = \beta + i\alpha\,, \;\; \lambda_4 = -\beta - i\alpha\,, \tag{20}$$

and in this case in follows from (19) for α and β

$$\alpha = \delta\sqrt{\omega/2}\left(1 + \sqrt{\left(1 - \frac{\nu^2}{4\delta^4}\right)}\right)^{1/2}, \quad \beta = \frac{\omega\nu}{4\alpha}\,. \tag{21}$$

In the case of small ν, retaining terms of the first order in ν, we have

$$\alpha = \delta\sqrt{\omega}\,, \;\; \beta = \frac{\nu\sqrt{\omega}}{4\delta}\,. \tag{22}$$

We distinguish from the general solution of Eq. (16)

$$X = \exp(i\omega t)\,(C_1 \exp(\lambda_1 z) + C_2 \exp(\lambda_2 z) + C_3 \exp(\lambda_3 z) + C_4 \exp(\lambda_4 z))$$

(C_i are the complex constants) its real part, having written it in the form

$$X = X_1(z)\cos\omega t + X_2(z)\sin\omega t\,. \tag{23}$$

Here the functions X_1, X_2 are the superposition of all possible products of the trigonometric and hyperbolic sine and cosine of arguments αz and βz. Presenting their set Φ in an ordered form

$$\Phi = \{\cos(\beta z)\,\mathrm{ch}(\alpha z)\,,\ \cos(\beta z)\,\mathrm{sh}(\alpha z)\,,\ \sin(\beta z)\,\mathrm{ch}(\alpha z)\,,\ \sin(\beta z)\,\mathrm{sh}(\alpha z)\,,$$
$$\cos(\alpha z)\,\mathrm{ch}(\beta z)\,,\ \cos(\alpha z)\,\mathrm{sh}(\beta z)\,,\ \sin(\alpha z)\,\mathrm{ch}(\beta z)\,,\ \sin(\alpha z)\,\mathrm{ch}(\beta z)\}\,,$$

we write ($n = 1, 2$)

$$X_n(z) = \sum_{i=1}^{8} P_{ni}^{(0)}\,\Phi_i(z)\,. \tag{24}$$

Among the sixteen coefficients presented here, only eight are independent. Let the coefficients in X_1 be them; the rest are related to them by the expressions

$$P_{21}^{(0)} = P_{14}^{(0)}\,,\;\; P_{22}^{(0)} = P_{13}^{(0)}\,,\;\; P_{23}^{(0)} = -P_{12}^{(0)}\,,\;\; P_{24}^{(0)} = -P_{11}^{(0)}\,,$$

$$P_{25}^{(0)} = P_{18}^{(0)}\,,\;\; P_{26}^{(0)} = P_{17}^{(0)}\,,\;\; P_{27}^{(0)} = -P_{16}^{(0)}\,,\;\; P_{28}^{(0)} = -P_{15}^{(0)}\,.$$

The values of $P_{1i}^{(0)}$ are found from the boundary conditions:

$$X_1 = X_2 = 0\,, \;\; X_1' = X_2' = 0 \quad (z = 0)\,,$$
$$X_1'' = X_2'' = 0\,, \;\; X_2''' = 0\,, \;\; X_1''' = f \quad (z = 1)\,. \tag{25}$$

The derivatives of X_n with respect to z can also be presented in the form (24). We have for the k-th derivative

$$X_n^{(k)}(z) = \sum_{i=1}^{8} P_{ni}^{(k)} \Phi_i(z)\,,$$

with the coefficients $P_{ni}^{(k)}$ being related by the recurrent relations:

$$P_{n1}^{(k)} = \alpha P_{n2}^{(k-1)} + \beta P_{n3}^{(k-1)}\,, \;\; P_{n2}^{(k)} = \alpha P_{n1}^{(k-1)} + \beta P_{n4}^{(k-1)}\,, \;\; P_{n3}^{(k)} = \alpha P_{n4}^{(k-1)} - \beta P_{n1}^{(k-1)}\,,$$

$$P_{n4}^{(k)} = \alpha P_{n3}^{(k-1)} - \beta P_{n2}^{(k-1)}\,, \;\; P_{n5}^{(k)} = \beta P_{n6}^{(k-1)} + \alpha P_{n7}^{(k-1)}\,, \;\; P_{n6}^{(k)} = \beta P_{n5}^{(k-1)} + \alpha P_{n8}^{(k-1)}\,,$$

$$P_{n7}^{(k)} = \beta P_{n8}^{(k-1)} - \alpha P_{n5}^{(k-1)}\,, \;\; P_{n8}^{(k)} = \beta P_{n7}^{(k-1)} - \alpha P_{n6}^{(k-1)}\,.$$

The boundary conditions are written in the form ($n = 1, 2$)

$$P_{n1}^{(0)} + P_{n5}^{(0)} = 0\,, \;\; P_{n1}^{(1)} + P_{n5}^{(1)} = 0\,, \;\; \sum_{i=1}^{8} P_{ni}^{(2)} \Phi_i(1) = 0\,,$$
$$\sum_{i=1}^{8} P_{2i}^{(3)} \Phi_i(1) = 0\,, \;\; \sum_{i=1}^{8} P_{1i}^{(3)} \Phi_i(1) = f\,, \;\; P_{15}^{(0)} = -P_{11}^{(0)}\,, \;\; P_{18}^{(0)} = -P_{14}^{(0)}\,. \tag{26}$$

Simplifying the notation ($P_{1i}^{(0)} = P_i$), we find for the coefficients $P_5 - P_8$ the relations:

$$P_5 = -P_1\,, \;\; P_6 = (A/C)\,P_3 - (B/C)\,P_2\,,$$

$$P_7 = -(A/C)\,P_2 - (B/C)\,P_3\,, \;\; P_8 = -P_4\,.$$

where $A = \alpha^2 - \beta^2$; $B = 2\alpha\beta$; $C = \alpha^2 + \beta^2$, and the coefficients $P_1 - P_4$ are found from the system of equations $M_{ij}P_j = N_i$ (where $N = (0, 0, 0, f)$):

$$M_{11} = A\Phi_1 - B\Phi_4 + A\Phi_5 + B\Phi_8\,, \;\; M_{12} = A\Phi_2 - B\Phi_3 + C\Phi_7\,,$$

$$M_{13} = B\Phi_2 + A\Phi_3 - C\Phi_6\,, \;\; M_{14} = B\Phi_1 + A\Phi_4 - B\Phi_5 + A\Phi_8\,,$$

$$M_{21} = -B\Phi_1 - A\Phi_4 + B\Phi_5 - A\Phi_8\,, \;\; M_{22} = -B\Phi_2 - A\Phi_3 + C\Phi_6\,,$$

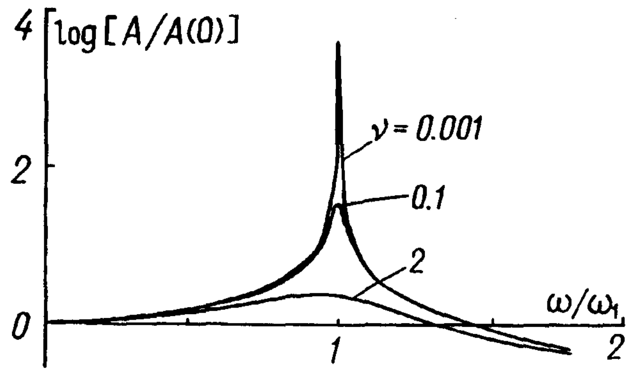
$$M_{23} = A\Phi_2 - B\Phi_3 + C\Phi_7\,, \;\; M_{24} = A\Phi_1 - B\Phi_4 + A\Phi_5 + B\Phi_8\,,$$

$$M_{31} = -A_1\Phi_2 - B_1\Phi_3 - B_1\Phi_6 - A\Phi_7\,, \;\; M_{32} = -A_1\Phi_1 - B_1\Phi_4 + \beta C\Phi_5 - \alpha C\Phi_8\,,$$

$$M_{33} = B_1\Phi_1 - A_1\Phi_4 + \alpha C\Phi_5 + \beta C\Phi_8\,, \;\; M_{34} = B_1\Phi_2 - A_1\Phi_3 + A_1\Phi_6 - B_1\Phi_7\,,$$

$$M_{41} = B_1\Phi_2 - A_1\Phi_3 + A_1\Phi_6 - B_1\Phi_7\,, \;\; M_{42} = B_1\Phi_1 - A_1\Phi_4 + \alpha C_1\Phi_5 + \beta C\Phi_8\,,$$

$$M_{43} = A_1\Phi_1 + B_1\Phi_4 - \beta C\Phi_5 + \alpha C\Phi_8\,, \;\; M_{44} = A_1\Phi_2 + B_1\Phi_3 + B_1\Phi_6 + A_1\Phi_7\,,$$



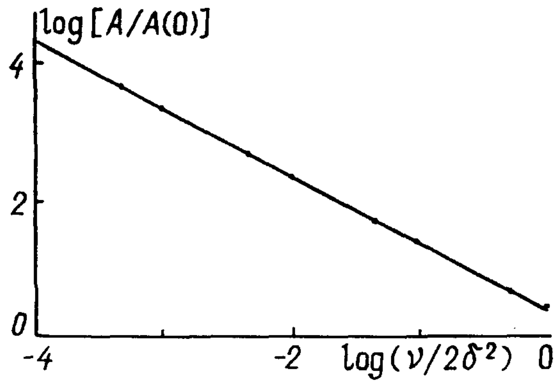


Fig. 1. Change in the amplitude-frequency characteristic of a three-layered rod with a liquid interlayer with an increase in the viscosity parameter ν ($\delta = 1$).

Fig. 2. Resonance amplitude A as a function of the viscosity parameter $\nu/(2\delta^2)$.

with $A_1 = \alpha A + \beta B$, $B = \alpha A - \beta B$, and the values of the functions $\Phi_i(z)$ are taken at the point $z = 1$.

We study the effect of the liquid interlayer on sandwich oscillations. According to (23), we write the time dependence of bending in the form

$$X(z, t) = X_0(z) \cos[\omega t - \varphi(z)],$$

where the amplitude and the phase of oscillations are determined according to

$$X_0 = (X_1^2 + X_2^2)^{1/2}, \quad \tan\varphi = \frac{X_2}{X_1}.$$

We consider the amplitude-frequency characteristic of the first mode of oscillations of the sandwich end $A(\omega)$, which is normalized over the amplitude on the zeroth frequency. The effect of the liquid interlayer is determined by the parameters δ and ν. The first characterizes the increase in mass at fixed elasticity and its growth decreases the eigenfrequency of oscillations. The parameter ν, characterizing viscosity, enters into the solution, according to (21), in the combination $\nu/(2\delta^2)$, whose maximum admitted value is equal to 1. The amplitude-frequency characteristic is shown in Fig. 1. Here the amplitude is referred to the deviation $A(0)$ of the sandwich end under the effect of a constant force, and frequency, to the eigenfrequency of the considered mode of oscillations of a rigid rod without losses (the value of this frequency in terms of T^{-1} is $\omega_1 = 3.52$ [6]). The dependence of the amplitude of oscillations on the value of the parameter $\nu/2\delta^2$ is presented in Fig. 2. As is seen, over practically the entire range of variation of this parameter an inversely propotional dependence takes place:

$$A/A(0) = 4\delta^2/\nu.$$

A small deviation is observed only near the limiting value $\nu/(2\delta^2) = 1$.

The value of liquid viscosity corresponding to the condition $\nu/(2\delta^2) = 1$ is determined according to ($\delta = 1$)

$$\eta = b\left(\frac{1}{3}\rho_r E\right)^{1/2}.$$

With the thickness of rods equal to 1 mm and the values of $\rho_r = 10$ g/cm^3, $E = 10^{12}$ dyne/cm^2 typical for metal, we find a value of η of the order of 10^5 poise. Thus, strong damping is possible when extremely viscous liquids are used. Extremely viscous liquids, however, possess elasticity with an ultimate time of relaxation of the stressed state [6]. The allowance for this factor may be the subject of a separate study.

NOTATION

l, b, h, length, width, and half-height of elastic rods; H, half-height of liquid layer; X, Y, Z and x, y, z, coordinates in systems associated with elastic rod and liquid layer; $\mathbf{n}$, $\boldsymbol{\tau}$, unit vectors of normal and tangent to liquid surface; V, velocity of longitudinal motion of point along surface of elastic rod; $\mathbf{v}$, P, η, ρ_f, velocity, pressure, viscosity, and density of liquid; σ_{ij}, tensor of viscous stresses; E, ρ_r, I, Young's modulus, material density, and inertial moment of cross-section area of elastic rods; S_r, S_f, areas of rod cross-section and liquid interlayer; A, ω, dimensionless amplitude and frequency of oscillations; δ, ν, dimensionless parameters characterizing increase in sandwich mass due to liquid and ratio of viscous to elastic forces. Subscripts: r, rigid; f, fluid.

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